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The Notions of Symmetry and Computational Feedback in the Paradigm of Steady, Simultaneous Quantum Computation

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Digital computation—i.e., the coherent concatenation of logical *if/then* statements—is generally mapped onto the temporal transformation of a physical state. In the alternative paradigm of steady, simultaneous quantum computation, logical concatenations are mapped onto the transformations of a quantum steady state into itself. Such transformations, separated from the time variable and thus freed from the one-way progression of time, can map *circular* logical concatenations. This gives rise to nondeterministic and nonrecursive computation. Toy model Hamiltonians of elementary (steady) computations are given to exemplify the applicability of the paradigm.

1. INTRODUCTION

We shall review the paradigm of steady simultaneous computation (Castagnoli, 1991; Castagnoli *et al.*, 1992) from a point of view which better highlights the importance of separating time from computation in the quantum framework.

In time-sequential computation, the state characterizing the process at a given time obviously cannot be influenced by its states at later times: in other words, as computation propagates from input to output, it cannot happen that the gate output contributes to the causal determination of the input of the same gate: computational feedback, or *circular computation*, is forbidden.[3]

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[3]One should notice that in time-sequential reversible computation, input is a function of the output (the input/output function is reversible), but this does not imply, of course, that output *causes* the input.

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We shall show that such a limitation is removed in the quantum framework when the computational process is *separable* from time, namely it can be mapped onto the transformations of a steady state into itself. Computation, seen as the mutual logical conditioning between gate input and/or output states, is mapped onto the branches and *loops* of an abstract network which represents the internal causality correlations (mutual and simultaneous) of the structure of a quantum steady state. This generates circular, and hence nondeterministic and nonrecursive, computation. Before describing in any detail such a model, it appears useful to review the basic notions concerning nondeterministic and nonrecursive computation, as well as time-sequential quantum computation.

There are classes of problems whose solution requires an amount of time or computer resources which grows exponentially with some measure of the problem size. Roughly, nondeterministic computation implies solving in polynomial time any problem of any of such classes. Whether or not Turing machines are capable of performing nondeterministic computation is a well-known open problem. Time-sequential quantum computation has—due to the fluctuations of computation time—the capability of overcoming the power of Turing machines (Deutsch, 1985; Brasher *et al.*, 1991), but computation time is the same as in Turing machines on average. Thus even though time-sequential quantum computation can provide some "nonclassical" advantage in competitive situations, such capability can be considered marginal.

We argue that the time-sequential paradigm may not be well suited to the exploitation of quantum laws. A propos of this latter observation, one should recall Feynman's (1986) criticism of time-sequential quantum computation: sequentiality is a *logical, not* a *physical* requirement, (time-sequential) quantum computation does not make much use of the specific qualities of the differential equations of quantum mechanics.

The notion of nonrecursivity is easily introduced in the framework of positive integer functions. By definition, nonrecursive functions are functions which are well defined (e.g., in second-order arithmetics), but are not Turing computable. While the Church thesis asserts that implementing nonrecursive computation is not possible, Penrose (1989) conjectures that there are physical situations, *well defined* in terms of physical laws, which are not Turing computable (see also Deutsch, 1985; Pour-El and Richards, 1983; Kreisel, 1974; Baez, 1983; Geroch and Hartle, 1986; among others).

2. REVIEWING THE NOTION OF STEADY, SIMULTANEOUS COMPUTATION

It is necessary to see first how some basic concepts inherent to the paradigm of classical computation transform themselves in the new paradigm. We shall introduce a definition of computation which holds in both paradigms:

- A **computation** process is the transformation of a physical state which maps the logical mathematical *definition of an object* into a physical state which maps the *object of the definition*, and is an observable from a physical standpoint.

For example, in classical computation, the initial state of a Turing machine implicitly *defines* the *result*, or the output of computation. Running the program provides the required transformation of the initial state into the output state, where the object of the definition is an observable.

It is useful to see the input–output transformation as a sequence of elementary transformations performed by logical gates (Fig. 1). Any gate is characterized by a *finite set of conditional statements*: if *input* $= \mathbb{I}_l$, then *output* $= \mathbb{O}_m$, where l ranges over all the possible input values and m is a function of l.

We shall consider sequential computation first [here gates are assumed to be reversible in the "conventional" fashion; see Bennett (1982) and Toffoli (1982)]. Its implementation requires the mapping of a chain of input/output values (*if*/*then* statements) onto the transformation undergone by a physical state under a temporal causal process (induced, e.g., by a suitable Hamiltonian).

However, we can think of mapping logical transformations onto other kinds of physical transformations, in particular onto a steady-state symmetry. By this we mean the transformation of a steady state vector—where the *time variable* has been *separated*—into itself (see Fig. 2). Notice that such a steady state would play both the roles of initial and final state of conventional computation, and therefore it should be *interpretable* both as definition of the object and as object of the definition, as required. In the sequel we will check that such a condition is verified.

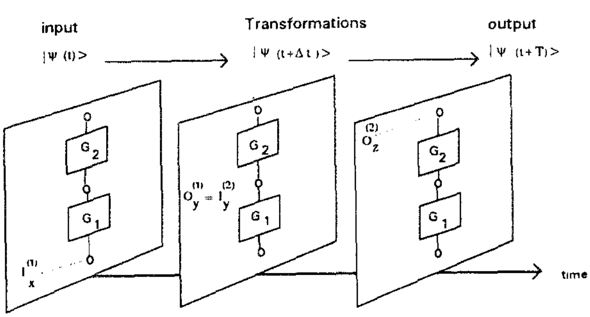

Fig. 1

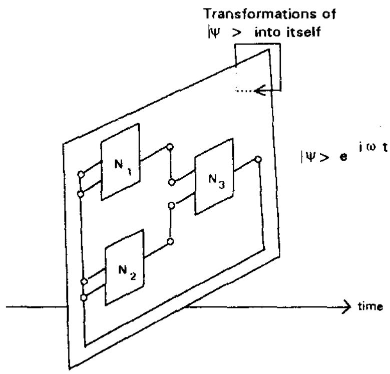

Fig. 2

Possible examples are the symmetries related to the transformations of a system of identical particles (bosons, fermions, anyons) into itself, which will be the case in the toy models discussed in Section 5. It is worth stressing that a quantum computation mapped onto such symmetries would comply with the reversibility property in the sense required by Feynman (1986), namely a reversibility requiring machine perfection or zero kelvin.

In fact, the inner symmetry properties of the state vector cannot change under any external perturbation; in other words, particle (or excitation) statistics cannot be changed. In this sense transformations of the system of particles into itself which does not alter the interaction and leaves the system *phase* (in the thermodynamic meaning) unaltered (we refer to this as quantum *stability*) can be considered perfect even at temperatures above zero kelvin.

3. STEADY, SIMULTANEOUS COMPUTATION IN FINITE AUTOMATA

We shall develop formally the model outlined in the case of finite automata, or Turing machines with a finite tape (Castagnoli, 1991). It is known that a finite automaton can be seen, without loss of generality, as an abstract machine which solves systems of Boolean equations. Such systems, incidentally, belong to the *exponential* class. Furthermore, any system of Boolean equations can be viewed as a system of NAND equations, which have some Boolean variables in common.

Figure 3a gives the truth table of the NAND equation, or gate, namely the set of conditional statements: *if* $a = 0$ **and** $b = 0$, *then* $c = 1$, etc. Note

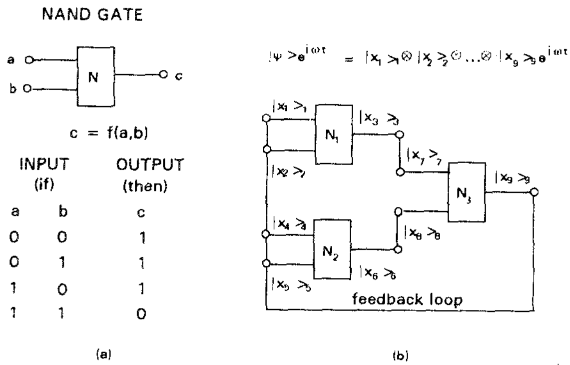

Fig. 3

that the NAND function $c = f(a, b)$ is not invertible; thus the input cannot be reconstructed from the output.

We will work on an example, whose generalization is straightforward. Consider the system of Boolean equations (where f is the NAND function):

$$\begin{aligned} &x_3 = f(x_1, x_2), \qquad x_6 = f(x_4, x_5), \qquad x_9 = f(x_7, x_8) \\ &x_3 = x_7, \qquad x_6 = x_8, \qquad x_1 = x_2 = x_4 = x_5 = x_9 \end{aligned} \tag{1}$$

Let us now connect together the inputs or outputs of any pair of gates which have some Boolean variable in common (see Fig. 3b). In a classical situation, this would generally "jam" the state of the network into an inconsistent configuration since it would in general *introduce loops of computational feedback*. Such loops require that the input of a gate *must be consistent with a function of its same output* (after some time delay). But a certain value of the output can admit different values of the input, given that the NAND function is not invertible. If the inputs are not chosen appropriately in the first place (but this is "exponentially improbable"), then inconsistency arises. According to present knowledge, the network of classical gates required to solve a system of n NAND equations requires a number of gates which is exponential in n (or time is exponential if a limited number of gates if reused).

We should remark that the process performed by a *classical* network of NAND gates is irreversible, since the temporal process goes from inputs to outputs, and in going from inputs to outputs information is destroyed (no input can be reconstructed from the output). Thus there is *destruction of information* along time. But when the process of transformations between gate inputs and outputs is *atemporal*, since time is separable, then a network of NAND gates can give a perfectly reversible process, as no information is destroyed along time. In our model, any eigenstate of the network of Fig. 3b should be the tensor product of a simultaneous set of *time-independent* eigenkets $|x_i\rangle_i$ multiplied by some phase factor $\exp\{i\omega t\}$.

We shall now introduce the quantum representation of such a reversible network. Any binary input or output—or node— is represented by a two-state "atom." For example, such states could be the two spin values of a fermion. This is represented formally by

$$\forall i: \quad \mathbf{n}_i|x_i\rangle_i = x_i|x_i\rangle_i \tag{2}$$

where $\mathbf{n}_i$ is the fermion number operator acting on Ket i. The generic state of the system of noninteracting atoms is thus given by

$$|\psi\rangle = \sum_{x_1,\ldots,x_9} w_{x_1,\ldots,x_9}|x_1\rangle_1 \cdots |x_9\rangle_9, \qquad \sum_{x_1,\ldots,x_9} |w_{x_1,\ldots,x_9}|^2 = 1 \tag{3}$$

with $x_1, \ldots, x_9$ assuming values in the set $\{0, 1\}$.

The tensor product sign is implied. Each tensor product appearing in the quantum superposition (3) is a combination of node eigenstates, i.e., of values of the inputs and outputs of all the gates of the network. Therefore such tensor products constitute a complete set of basis vectors for representing the state of the network, of course even when the degrees of freedom of the network are limited by the action of the gates. Such basis vectors span a Hilbert space $\mathscr{H}$. A NAND gate, say N_1, is represented by equation (4), which projects $\mathscr{H}$ upon the subspace $\mathscr{H}_1$ spanned by the basis vectors (each a combination of input output values) compatible with the NAND equation in question, as can be readily seen:

$$\mathbf{N}_1|\psi\rangle = |\psi\rangle \tag{4}$$

where

$$\mathbf{N}_1 = \sum_{x_1,x_2} |x_1\rangle_1|x_2\rangle_2|f(x_1, x_2)\rangle_3\langle f(x_1, x_2)|_3\langle x_2|_2\langle x_1|_1$$

Similarly, when two input or output variables coincide, say $x_3 = x_7$, this is represented by equation (5), which projects $\mathscr{H}$ upon $\mathscr{H}_{3,7}$ spanned by the basis vectors compatible with the Boolean equation $x_3 = x_7$:

$$\mathbf{U}_{3,7}|\psi\rangle = |\psi\rangle, \qquad \mathbf{U}_{3,7} = |0\rangle_3|0\rangle_7\langle 0|_7\langle 0|_3 + |1\rangle_3|1\rangle_7\langle 1|_7\langle 1|_3 \tag{5}$$

In this way any Boolean equation is associated with an operator equation. The simultaneous system of such operator equations,

$$\begin{aligned} &\mathbf{N}_\alpha|\psi\rangle = |\psi\rangle, \qquad \alpha = 1, 2, 3 \\ &\mathbf{U}_{3,7}|\psi\rangle = |\psi\rangle, \qquad \mathbf{U}_{6,8}|\psi\rangle = |\psi\rangle \\ &\mathbf{U}_{1,9}|\psi\rangle = \mathbf{U}_{2,9}|\psi\rangle = \mathbf{U}_{4,9}|\psi\rangle = \mathbf{U}_{5,9}|\psi\rangle = |\psi\rangle \end{aligned} \tag{6}$$

projects $\mathscr{H}$ upon the intersection of all the subspaces generated by the individual equations. Notice that the operators $\mathbf{N}, \ldots, \mathbf{U}, \ldots$ are projectors. It is understood that equations (6) comprise equations (3) and (2), namely the definition of the generic $|\psi\rangle$ and of the eigenvalues x_i. The intersection in question is spanned by the basis vectors compatible with all the equations; each of such vectors represents a combination of input–output values compatible with the network of NAND gates, and therefore a solution of the system of Boolean equations. Equations (6) define a quantum object in the following abstract sense [notice the analogy with Albert's (1983) automata]:

1. Any state of the object is a vector of a Hilbert space.
2. Any linear combination of such vectors represents a state of the object.
3. All measurable properties of the object correspond to a set of commuting observables on that Hilbert space.

All of this can be checked by noticing that equations (6) can be written in the form

$$\mathbf{P}|\psi\rangle = m|\psi\rangle, \qquad \text{where } \mathbf{P} = \mathbf{N}_1 \cdot \mathbf{N}_2 \cdot \ldots \cdot \mathbf{U}_{3,7} \cdot \mathbf{U}_{6,8} \cdot \ldots; \quad m = 1 \tag{7}$$

The fermion number operator $\mathbf{n}_i$ and the projector $\mathbf{P}$ are all diagonal and therefore pairwise commutative: their eigenvalues are simultaneously measurable. In the preparation $m = 1$, a set of simultaneous eigenvalues of $\mathbf{n}_i$ maps a consistent set of input–output values of the gates (a solution). There can be multiple solutions in quantum superposition.

We have pointed out that the network of Figure 3b is reversible, since time is assumed to be a separable variable. For this reason we say not that the unwanted amplitudes are canceled (such a statement would imply a *before* and an *after*) but that they are *steadily kept canceled.*

The computational model described above lends itself to a number of comments.

1. It is an algebraic *asequential* form of computation. This may be better understood by means of the following consideration. We have a system of equations operating on the state vector, namely on a linear combination of basis vectors. Let us assume that quantum description is *objective* in character. By this, intuitively, we mean that the operator equations are objective physical bounds acting on the amplitudes of the basis vectors which are the relevant *objective* physical entities. In this conceptual model, the amplitudes of the basis vectors are the *unknowns* of the computational problem, which is thus solved by physically keeping to zero all the unwanted amplitudes.

2. In a computational model where the solution is obtained by keeping canceled the unwanted amplitudes, the same state can in fact be interpreted simultaneously as:

(a) The object of the definition, namely the result of computation: since the unwanted amplitudes are canceled, the state maps the result of computation.
(b) The computation: to recognize such a state as a computation, it is sufficient to think that all the amplitudes are entered into the system of simultaneous equations as the unknowns of the computational problem, while the transformations of the state into itself keep the unwanted amplitudes canceled.
(c) The definition of the object: the system of simultaneous operator equations, besides defining a physical state, maps exactly the definition of the result, namely the system of Boolean equations.

One can say that the object (the steady state) is the definition and the computation of itself. The notion that a steady state is, circularly, the *physical cause* of itself appears then as natural, since it is but the application of the usual principle of physical (temporal) causality to steady states. The computational interpretation of this notion is what leads to the concept of circular—*nondeterministic* computation.

(3) We discuss now the notion of computational feedback. Let us call *causality* the transformations of the network state into itself, established by the network of quantum NAND gates, namely by equations (6). In the limiting case of steady states the character of causality is modified, to become *atemporal* and *mathematically simultaneous*. The latter property has the conventional meaning: the transformation established, say, by the equation $|\psi\rangle = \mathbf{N}_1 |\psi\rangle$, between the steady-state vector $|\psi\rangle$ and itself is mathematically simultaneous (trivially, such a transformation does not involve the notion of a temporal propagator from $|\psi\rangle$ to $|\psi\rangle$ itself!). More generally, equations (6) establish a system of simultaneous transformations between $|\psi\rangle$ and itself.

Such transformations can be alternatively interpreted as the mutual simultaneous transformations between the network input–output states (whose tensor product gives, in fact, the compound state $|\psi\rangle$). Thus it is legitimate to interpret such (objective) simultaneous transformations as mutual simultaneous *causation* of the network input–output states. One can therefore state that simultaneous causality can be viewed as the quantum togetherness of causally compatible states. This is of course a strictly quantum notion, since it implies the possibility that causality, or computation, is circular. In fact, it is circular along the network loops which comprise a gate input–output such that the input is a function of the output (see, e.g., Fig. 3b).

As a consequence of simultaneous computation, the state of *each* node must be simultaneously and globally consistent with the states of *all* nodes, according to the transformations defined by the network of quantum NAND gates. This means just "nondeterministic computation." It transforms problems of *exponential complexity* in the framework of *classical* computation, which require the *temporal interaction of a number of "atoms" (binary inputs or outputs) of the order of* 2^n, into problems of *polynomial complexity* in the still abstract *quantum* framework, requiring the *simultaneous interaction of n "atoms."*

One could say that nondeterministic computation is just an instance of quantum togetherness, more precisely of the causality togetherness inherent to a steady quantum structure.

4. It is worth briefly discussing how conventional computation, namely the overall input–output relationship, can be mapped onto a network of quantum NAND gates. The nature of such a mapping is the following. The overall input–output pair is mapped onto a pair of subsets of all possible network inputs and outputs. Input values should not be thought of as set from an agent outside the network: the NAND function is a universal primitive and a simultaneous system of such primitives (part of the network itself) can represent any constant function, e.g., $x_i = 1$, where x_i would be an input variable. (Or a different network can be a superposition which maps multiple input–output pairs.) Thus the problem of setting the input can be included in the more general problem of implementing similar networks.

5. The model of steady simultaneous computation can be extended to the computation of hyperarithmetical functions, a class which comprises all the recursive functions as well as many nonrecursive ones (Castagnoli *et al.*, 1992; Castagnoli and Vincenzi, 1991).

5. TOY MODEL HAMILTONIANS

We shall now briefly discuss two toy model Hamiltonians, which exemplify the practical applicability of the paradigm of steady simultaneous computation, and at the same time highlight the relationship between steady computation and quantum symmetry.

The first is the simplified version of the Hubbard (1963, 1964) model referred to as the Falikov and Kimball (1969) model, over a lattice Λ consisting of only two sites, **1** and **2**, whose Hamiltonian can be written as

$$\mathscr{H} = \sum_{\mathbf{i}\in\Lambda} \epsilon_{\mathbf{i}} N_{\mathbf{i}} - t_A(A_1^\dagger A_2 + A_2^\dagger A_1) + C \tag{8}$$

with

$$\epsilon_{\mathbf{i}} \equiv -\mu_A + UD_{\mathbf{i}}; \qquad C \equiv -\mu_B \sum_{\mathbf{i}\in\Lambda} D_{\mathbf{i}} \tag{9}$$

The variables $A_{\mathbf{i}}$ are (annihilation operators) for *spinless* fermions—related to the customary annihilation operators for fermions with spin $\sigma \in \{\Uparrow, \Downarrow\}$ at site $\mathbf{i}$ of Λ by $A_{\mathbf{i}} \equiv (1/\sqrt{2})(a_{\mathbf{i},\Uparrow} + a_{\mathbf{i},\Downarrow})$—whereas $N_{\mathbf{i}} \equiv A_{\mathbf{i}}^{\dagger} A_{\mathbf{i}}$. Moreover, the variables $D_{\mathbf{i}}$ are the number operators $D_{\mathbf{i}} \equiv B_{\mathbf{i}}^{\dagger} B_{\mathbf{i}}$ for the spinless fermions $B_{\mathbf{i}} \equiv (1/\sqrt{2})(a_{\mathbf{i},\Uparrow} - a_{\mathbf{i},\Downarrow})$. The $D_{\mathbf{i}}$ are idempotent ($D_{\mathbf{i}}^2 = D_{\mathbf{i}}$, $\forall \mathbf{i}\in\Lambda$) and *central* i.e., they commute with all dynamical variables in the system (functions only of the $A_{\mathbf{i}}$). In other words, they are constants of motion, unaffected by the dynamics, which can be thought of as *classical* variables ($D_{\mathbf{i}} = 0, 1$): their role is to split the Hilbert space of states of the system into four orthogonal *sectors*, each corresponding to one of the possible values of the *pair* $\{D_1, D_2\}$. Finally, μ_A, μ_B are the chemical potentials for particles of type A and B respectively, t_A denotes the hopping amplitude for A particles, and U is the *effective* local Coulomb interaction strength.

The Hamiltonian $\mathscr{H}$ can be diagonalized—after noticing that it has a *dynamical algebra* $u(1) \oplus su(2)$, as one can check by writing it in the form

$$\mathscr{H} = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2)J + (\varepsilon_1 - \varepsilon_2)K_z - t_A(K_+ + K_-) + C \tag{10}$$

where $J \doteq (N_1 + N_2)$ generates $u(1)$ (J represents the conserved total number of A particles), whereas $K_+ \doteq A_1^{\dagger} A_2$, $K_- \doteq A_2^{\dagger} A_1 = K_+^{\dagger}$, $K_z \doteq \frac{1}{2}(N_1 - N_2)$ generate $su(2)$ ($[K_z, K_\pm] = \pm K_\pm$, $[K_+, K_-] = 2K_z$)—by a unitary transformation [a rotation in the group manifold of $SU(2)$] mapping it onto a linear combination of J and K_z. The diagonal form of (8) is given by

$$\begin{aligned}
\mathscr{H}_{\text{diag}} &\equiv \exp\{\text{ad } Z\}\ (\mathscr{H}) \\
&\doteq \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{[Z, [Z, \ldots [Z, \mathscr{H}] \ldots]]}_{n \text{ brackets}} \\
&= \frac{1}{2}(\varepsilon_1 + \varepsilon_2)J + [(\varepsilon_1 - \varepsilon_2)^2 + 4t_A^2]^{1/2} K_z + C \\
&\equiv \varepsilon_+ N_1 + \varepsilon_- N_2 + C
\end{aligned} \tag{11}$$

with

$$\begin{aligned}
&Z \doteq \vartheta(K_+ - K_-), \qquad \vartheta = -\frac{1}{2}\tan^{-1}\frac{2t_A}{(\varepsilon_1 - \varepsilon_2)} \\
&\varepsilon_\pm = \frac{1}{2}\{(\varepsilon_1 + \varepsilon_2) \pm [(\varepsilon_1 - \varepsilon_2)^2 + 4t_A^2]^{1/2}\}
\end{aligned} \tag{12}$$

The time evolution of a generic initial state $|\psi_0\rangle$ is generated by the unitary operator $\mathscr{U} \doteq \exp(it\mathscr{H})$,

$$|\psi_t\rangle = \mathscr{U}|\psi_0\rangle = e^{Z^\dagger} e^{\mathrm{ad}\, Z}(\mathscr{U}) e^{Z}|\psi_0\rangle \tag{13}$$

Representing the initial state over the Fock space as $|\psi_0\rangle \equiv \sum_{n_1,n_2} \zeta_{n_1,n_2}|n_1 n_2\rangle$, we find

$$|\psi_t\rangle = e^{itC}\{\tilde{\zeta}_{01}|01\rangle + \tilde{\zeta}_{10}|10\rangle\} \tag{14}$$

where $\tilde{\zeta}_{01}$ and $\tilde{\zeta}_{10} = (1 - \tilde{\zeta}_{01}^2)^{1/2}$ are proportional to the corresponding ζ_{n_1,n_2} through coefficients—functions of the parameters t_A, μ_A, μ_B, U, and $\{D_\mathrm{i}|\mathrm{i} = 1, 2\}$—which can be made time independent by a suitable choice of the distribution of these latter. In this case, equation (14) manifests the feature that, whatever the initial state, the Hamiltonian (8) induces a stationary state which is a quantum superposition of pure states which are solutions of the Boolean function NOT:

$\vert a\rangle$	$\vert b\rangle = f(\vert a\rangle)$
$\vert 0\rangle$	$\vert 1\rangle$
$\vert 1\rangle$	$\vert 0\rangle$

(where of course $|n_1 n_2\rangle \Rightarrow |a\rangle|b\rangle$), in which the states with $n_1 = n_2$, i.e., $|00\rangle$ and $|11\rangle$, are *projected off*. A suitable preparation, or interaction from the outside, could allow us to *keep canceled*, as required, either one of such two amplitudes, which is what is required for a "good" NOT gate, and thus lead to the possibility of networking together a system of gates.

The second model is defined by the *abstract* Hamiltonian over a one-dimensional lattice ambient space with n sites and periodic boundary conditions:

$$H = \varepsilon \sum_{\mathrm{i}=1}^{n} B_\mathrm{i} + \mathscr{W} \sum_{\{\pi(\mathrm{i}) \in \mathscr{P}_n | \mathrm{i} = 1,\ldots,n\}} B_{\pi(1)} \cdots B_{\pi(\mathrm{n})} = \mathscr{W}' \sum_{\substack{\mathrm{i}=1 \\ \mathrm{mod}\, n}}^{n} B_\mathrm{i} B_{\mathrm{i}+1} B_\mathrm{i} \tag{15}$$

where $\mathscr{P}_n$ denotes the set of permutations of n objects, and $\{B_\mathrm{i}|\mathrm{i} = 1, \ldots, n\}$ are the generators of the *braid group* (Birman, 1982) algebra $\mathscr{B}_n$, with defining relations

$$B_\mathrm{i} B_\mathrm{j} = B_\mathrm{j} B_\mathrm{i} \qquad \text{for} \quad |\mathrm{i} - \mathrm{j}| \geq 2 \tag*{[i]}$$

$$B_\mathrm{i} B_{\mathrm{i}\pm 1} B_\mathrm{i} = B_{\mathrm{i}\pm 1} B_\mathrm{i} B_{\mathrm{i}\pm 1}, \qquad \mathrm{i} = 2, \ldots, n-1 \tag*{[ii]}$$

together with the constraint

$$B_\mathrm{i}^2 = (q - q^{-1})B_\mathrm{i} + 1, \qquad \mathrm{i} = 1, \ldots, n \tag*{[iii]}$$

where $q \in \mathbb{R}$ is an arbitrary parameter.

$\mathscr{B}_n$ has a physically interesting realization in terms of the customary fermionic creation and annihilation operators ($n_{\mathrm{i}} \equiv a_{\mathrm{i}}^\dagger a_{\mathrm{i}}$):

$$B_{\mathrm{j}} = (a_{\mathrm{j}}^\dagger a_{\mathrm{j}+1} + a_{\mathrm{j}+1}^\dagger a_{\mathrm{j}}) - (qn_{\mathrm{j}} + q^{-1}n_{\mathrm{j}+1}) + q, \qquad \mathrm{j} = 1, \ldots, n \quad (16)$$

From the latter, upon defining first $\tau_{\mathrm{j}}^{(x)} \doteq (a_{\mathrm{j}}^\dagger + a_{\mathrm{j}})$, $\tau_{\mathrm{j}}^{(p)} \doteq -i(a_{\mathrm{j}}^\dagger - a_{\mathrm{j}})$, and successively performing on the $\{\tau_{\mathrm{j}}^{(x,p)}\}$ the Jordan–Wigner transformation

$$\tau_{\mathrm{j}}^{(x,p)} = \exp\left[i\frac{\pi}{2}\sum_{\mathrm{k}=1}^{j-1} (\sigma_{\mathrm{k}}^{(0)} + 1)\right]\sigma_{\mathrm{j}}^{(x,p)} \qquad (17)$$

where

$$\sigma_{\mathrm{i}}^{(\alpha)} \doteq \overbrace{\mathbb{I} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes \underbrace{\sigma^{(\alpha)}}_{\text{ith place}} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}}^{n \text{ factors}}, \qquad \alpha = x, p, 0 \quad (18)$$

the $\sigma^{(\alpha)}$ are the customary Pauli matrices, defining a spin-1/2 representation of $su(2)$, we are led to another realization of $\mathscr{B}_n$, in terms of *spins*:

$$B_{\mathrm{j}} = \tfrac{1}{2}[(\sigma_{\mathrm{j}}^{(x)}\sigma_{\mathrm{j}+1}^{(x)} + \sigma_{\mathrm{j}}^{(p)}\sigma_{\mathrm{j}+1}^{(p)}) - (q\sigma_{\mathrm{j}}^{(0)} + q^{-1}\sigma_{\mathrm{j}+1}^{(0)})] \qquad (19)$$

The Hamiltonian H—which can thus be interpreted either as that of an interacting *anyon* system or of an interacting *fermionic excitation* system, or yet as the Hamiltonian of an interacting *spin* system—can be diagonalized thanks to the property of being invariant under the *quantum supergroup* (Drin'feld, 1985; Manin, 1989; Woronowicz, 1987) $su(1|1)_q$. Denoting by Δ the coproduct operation characteristic of the latter, and defining

$$\mathbf{T}^{(\alpha)} \doteq \Delta^n(\tau^{(\alpha)}) \equiv q^{(n+1)/2} \sum_{\mathrm{j}=1}^{n} q^{\mathrm{j}}\tau_{\mathrm{j}}^{(\alpha)}, \qquad \alpha = x, p \qquad (20)$$

{generating $su(1|1)_q$ by $\{\mathbf{T}^{(\alpha)}, \mathbf{T}^{(\alpha)}\} = 2[n]_q$, $\{\mathbf{T}^{(x)}, \mathbf{T}^{(p)}\} = 0$, where, as usual, we denote $[n]_q \doteq (q^n - q^{-n})/(q - q^{-1})$}, one can check that indeed

$$[\mathbf{T}^{(x)}, H] = 0 = [\mathbf{T}^{(p)}, H] \qquad (21)$$

We referred above to the system defined by H as *anyonic* just for this reason.

We assume now $n = 3$. A lengthy but relatively standard calculation gives for the time evolved $|\psi_t\rangle \doteq e^{itH}|\psi_0\rangle$ of an arbitrary initial state, assigned—in the fermonic representation—as

$$|\psi_0\rangle = \sum_{n_1, n_2, n_3} \xi_{n_1,n_2,n_3}|n_1 n_2 n_3\rangle \qquad (22)$$

the form

$$|\psi_t\rangle = e^{it\omega}(\tilde{\xi}_{000}|000\rangle + \tilde{\xi}_{010}|010\rangle + \tilde{\xi}_{100}|100\rangle + \tilde{\xi}_{111}|111\rangle) \qquad (23)$$

where once more the coefficients$\{\tilde{\xi}_{n_1,n_2,n_3}\}$ are proportional to the ξ_{n_1,n_2,n_3}

through functions of ε, $\mathscr{W}$, $\mathscr{W}'$ (obviously constrained by the condition of normalization of $|\psi_t\rangle$), and frequency ω is *constant* (in time) *iff* $\mathscr{W}$ is an assigned function of $\mathscr{W}'$. There follows that a Hamiltonian of the form (18) with a suitable choice of the *control parameters* may induce the stationary realization of the Boolean function AND:

$\|a\rangle$	$\|b\rangle$	$\|c\rangle = f(\|a\rangle, \|b\rangle)$
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$
$\|1\rangle$	$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$

($|n_1 n_2 n_3\rangle \Rightarrow |a\rangle|b\rangle|c\rangle$). Once more notice that the amplitudes $\xi_{000}, \ldots, \xi_{111}$ are time independent.

A possible task for further work is of course to network together AND and NOT gates. This would require representing by suitable coherent interactions the equality between Boolean variables common to different gates.

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